Critical Discussions

A math pedagogy designed to empower learners

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When I ask math students about what it means to do math, I typically get ideas that fall into two general categories. The most common response, particularly across Grades 7 to 12, is that math is about getting the right answer as quickly as possible. The second most common response comes more frequently from younger students. They generally say that math is about solving a problem in a way they most like or is easiest for them. Of course, students also say that it means they get to work in groups, use computers, write on walls or desks, or choose questions that are "right" for them. On rare occasions, a student says that math is about thinking! Despite these varying responses, the two most common, and interestingly diametrically opposed, responses have led math educators to wonder whether there is a more balanced, enriched way for students to learn math that draws on the benefits of both approaches and empowers all students to be successful in math.

Years have been spent debating how to effectively help students learn mathematics. Many math educators have offered diverse suggestions on how to make this happen. At one end of the pendulum swing is the continued support for the century-old practice of teaching math in a procedural manner[1] in which students memorize content and algorithms and repeatedly mimic what the educator is modelling. On the other end is support for an approach in which students are given problems to solve that require content knowledge and understanding that they do not yet possess in the hope that they will construct it through discovery and collaboration.

Unfortunately, neither of these approaches has empowered the vast majority of students with the capacity to be successful in math and to see its relevance in their world.[2] Instead, math educators have been pushed to take a stance in support of one approach or the other, with minimal positive effect on student learning. A more balanced, comprehensive approach is needed if we are to change the teaching and learning landscape of mathematics in a positive, less contentious way. A more viable and powerful solution is for students to learn to reason proficiently using critical thinking; an approach that also increases the likelihood of students automatizing math content.

[1] Schoenfold, Alan, A. (2016). *Research in Mathematics Education.* Review of Research in Education, 40(1), pages 497–528. https://doi.org/10.3102/0091732X16658650

[2] The most recent PISA results show that on average only 15% of Canadian and 11% of students across OECD countries achieve proficiency at the top two level of achievement combined. See Deussing, M., Fung, K., O'Grady, K., Scerbina, T., Muhe, N. (2019). *Measuring up: Canadian Results of the OECD PISA 2018 Study. The Performance of Canadian 15-Year-Olds in Reading, Mathematics, and Science.* The Council of Ministers of Education, Canada https://www.cmec.ca/Publications/Lists/Publications/Attachments/365/PISA2015-CdnReport-EN.pdf

MATHEMATICAL REASONING THROUGH CRITICAL THINKING

The development of mathematical reasoning competencies paired with critical thinking harnesses the benefits of a wide range of mathematical learning approaches while improving every student's capacity for success in math. This pedagogical approach ensures that students develop *both*

1. a strong understanding of foundational math concepts and content that they can retain over time and immediately access when needed, and
2. the capacity to reason soundly about and with these concepts and content so that they can be flexibly applied to solve any problem, in any context, for any purpose, and for any audience.

With this approach, students learn important math content and how to deeply **understand** this content, appropriately **act** on it, and effectively **communicate** it to achieve overall learning success in math, as illustrated in Figure 1.

THE SYNERGY BETWEEN CRITICAL THINKING AND MATHEMATICAL REASONING COMPETENCIES

There is an important relationship between a student's ability to reason soundly and the quality of their understanding, actions, and communication in math. Students may only understand, act, and communicate effectively in math if they pay close attention to the quality of their reasoning. In other words, students need to think critically, assessing their understanding, actions, and communications in light of criteria that clearly define the quality of each. A reasoning competency approach to math learning ensures that students understand the importance of applying sound reasoning to all aspects of their learning in order to be successful in math.

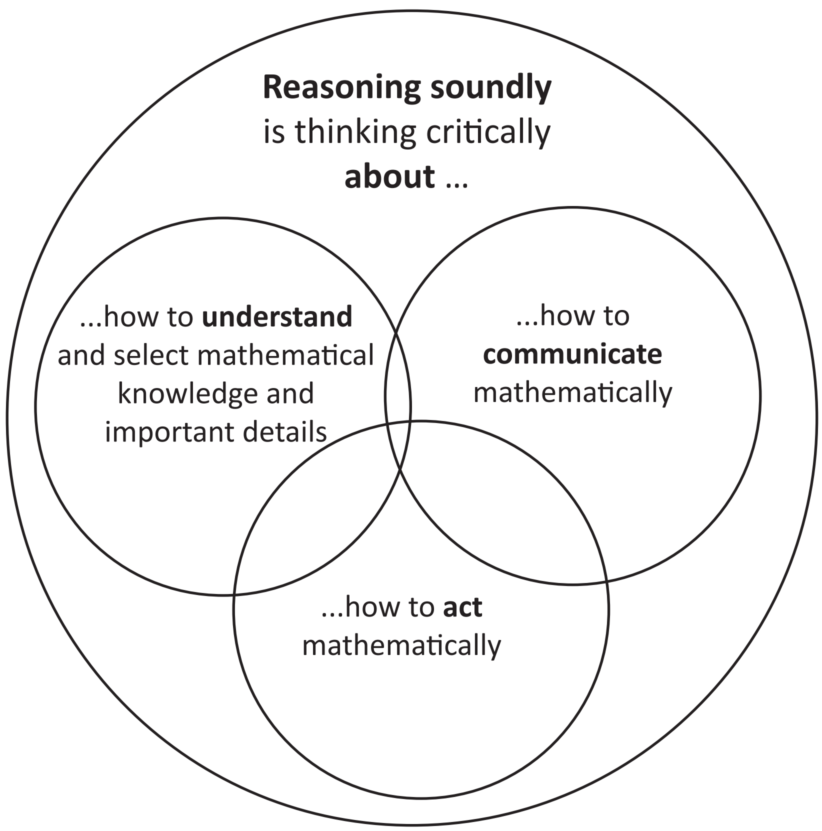

Figure 1: Sound Reasoning
Every aspect of mathematics requires a person to reason soundly; the ability to do so shapes how well they are able to understand, act, and communicate mathematically.

Mathematical Reasoning Competencies

A math learner needs to possess a broad range of reasoning abilities to understand, act, and communicate in a way that allows them to effectively solve challenging, often unfamiliar, and problematic tasks across a range of contexts, situations, and audiences involving mathematics.[3] A review of the mathematical skills and **reasoning competencies** identified in a range of research and curriculum documents from around the world[4] found that there are 8 key mathematical reasoning competencies needed for success in math, as outlined in Table 1.

The first, *sound reasoning* is one of the core reasoning competencies upon which all mathematical abilities are built. The *quality* of all other mathematical reasoning competencies depends strongly on the capacity to think in a deliberate and sound manner. Sound reasoning ensures that all other forms of reasoning are of high quality. The second core reasoning competency, *reflective reasoning* ensures that all other forms of reasoning are continuously and iteratively monitored, assessed, and modified for their quality. It also fluidly and flexibly connects all of the other seven forms of reasoning. Together there are two forms of reasoning necessary for the development of rich, automatized understandings, two forms necessary for the selection of effective mathematical actions, and two forms necessary for effective mathematical communication. See Figure 2 for the connections among the reasoning competencies.

Table 1: 8 Key Mathematical Reasoning Competencies

Competency	Definition
Core competencies	
Sound thinking or reasoning	Reasoning about the quality of one's thinking
Reflective reasoning	Reasoning about the quality of one's understanding, actions, and communication
Understanding	
Conceptual reasoning	Reasoning about what makes a concept what it is and how to recognize that concept
Detail-minded reasoning	Reasoning about which mathematical details to use and how to use them effectively
Acting	
Connective reasoning	Reasoning about how to effectively connect ideas
Problem-managing reasoning	Reasoning about how to effectively identify and solve math problems (including how to effectively select, organize, and use mathematical strategies, tools, and resources)
Communicating	
Representational reasoning	Reasoning about how to effectively represent ideas
Language reasoning	Reasoning about how to effectively use the language of mathematics including its structure and symbols

[3] OECD. (2018). PISA 2021 Mathematics Framework (DRAFT). https://pisa2021-maths.oecd.org/

[4] Sources used to develop a synthesized set of 8 key mathematical reasoning are found in the appendix.

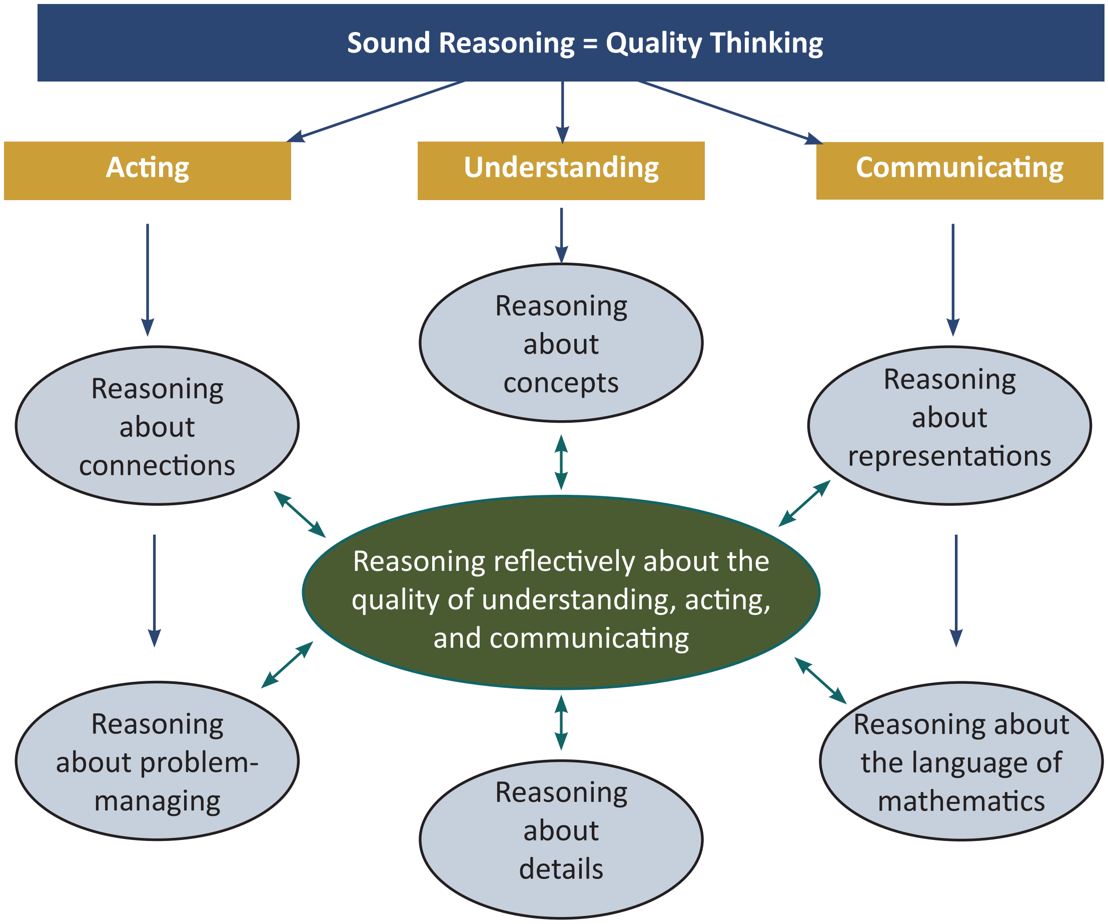

Figure 2: The Interconnectedness of the Reasoning Competencies
Successful student learning of mathematics depends on the capacity to reason mathematically in a number of important and interconnected ways.

Critical Thinking in Mathematics

To reason soundly in mathematics, students need to learn how to think critically, that is, to make reasoned mathematical decisions. The study of mathematics is full of decision-making. To do well in math, student decision-making may include

- determining the true, generally acceptable, meaning of mathematical ideas (conceptual reasoning);
- what knowledge or understanding to draw upon that is relevant and useful to identifying and solving a problem (problem-managing reasoning);
- how to best represent a mathematical idea (representational reasoning); or
- to what degree one idea connects to others (connective reasoning).

Of course, mathematics can be taught without any decision-making if someone tells students what to do every step of the way. But this does not help students develop their abilities to think independently, problem-manage, and apply concepts to other situations.

This leaves math educators with one important question: *how do students learn to make these decisions and learn to make them well?*

The answer is: to make sound mathematical decisions, students need to consider and assess the mathematical options they have available to them so that they can choose the best one(s). They do this by thinking about the considerations or attributes—*the criteria*—that define "best" in a given context and then use these criteria to guide their decision-making.

To provide each and every student equitable opportunities to improve their learning success in math, students need to learn how to reason soundly in a variety of ways through the application of critical thinking.

Empowering Students with Sound Reasoning

To illustrate how students learn to make sound mathematical decisions, let's consider an example. A class of Grade 4 students is asked to decide which operation, addition or multiplication (or both), is best (most appropriate) to use to calculate the perimeter of a play space they are tasked with redesigning. To successfully make a sound mathematical decision, students must apply the criteria for sound reasoning, as outlined in Table 2.

Clearly students can only make sound mathematical decisions if they possess the required mathematical knowledge and understanding to do so, but they must also explicitly think about how their understanding and knowledge fit with the relevant mathematical details provided, or assumed, in the context of the problem. Explicitly providing students with criteria for sound reasoning ensures that every student can come to understand what is for many an elusive way of effectively reasoning through a math question or task. Once students are introduced to the mathematical decision they need to make, it is important to either review or construct their knowledge and understanding by having them develop and apply the relevant reasoning competencies in a sound way.

For example, in the Grade 4 lesson, students needed to apply

- a strong conceptual understanding of the true (generally accepted) meaning of addition, multiplication, and perimeter through *conceptual reasoning*, and
- the relationship between these three concepts through *connective reasoning*.

As students learn math, they need to draw upon a range of relevant reasoning competencies if they are to experience success.

For reasoning to be sound in mathematics, students must ensure that any decision, choice, or conclusion they make is

- consistent (fits) with what they already know and understand to be mathematically true, and
- consistent (fits) with the relevant mathematical details provided, or assumed, when necessary.

Table 2: Criteria for Sound Reasoning for a Grade 4 Lesson

<table>
<tr><th colspan="2">Criteria for Sound Reasoning in Action</th></tr>
<tr><td>(1)
Decision is consistent (fits) with what students already know and understand to be mathematically true about …</td><td>(2)
Decision is consistent (fits) with the relevant mathematical details provided, or assumed, in the context of …</td></tr>
<tr><td>… how to calculate perimeter.

For example, students must know and understand
• the attributes of the geometric shapes they are measuring;
• the meaning of perimeter: the total of the side lengths of a given shape;
• the meaning of addition: the total of two or more values; and
• the meaning of multiplication: repeated addition of the same value.</td><td>… choosing the best operation, multiplication or addition (or both).

For example, the student decision
• must be consistent (fit) with the properties of geometric shapes relevant to the concept of perimeter; in this case, the information on side lengths of the geometric shape of their play space.</td></tr>
</table>

Table 3: Three-part Lesson Structure

Specific Grade 4 Lesson Example	General Process
Part One: *Start Student Thinking* Invite each student to initially decide and justify which operation, addition or multiplication (or both), is best to use to calculate the perimeter of a given shape based on everything they already know and understand.	**Part One:** *Start Student Thinking* Invite students to independently make a conjecture or a prediction or begin responding to the lesson question or task based on everything they already know and understand mathematically.
Part Two: *Grow Student Thinking* Strengthen and connect what students already know and understand about addition, multiplication, and perimeter by investigating the perimeters of various shapes: some with no equal side lengths, some with a few equal side lengths, and others with all equal side lengths. This is most effectively achieved in collaboration with their peers.	**Part Two:** *Grow Student Thinking* Deepen or clarify past learning or create new learning required to complete the lesson question or task by providing students with a collaborative opportunity to investigate ideas in a variety of contexts through a series of authentic examples that move from the concrete and practical to the more general and abstract.
Part Three: *Pull Together Student Thinking (Consolidate)* Invite students to decide on how these two operations are related in the context of calculating perimeter and to use reflective reasoning to decide whether they should affirm, modify, and/or extend their initial thinking at the start of the lesson based on their new learning.	**Part Three:** *Pull Together Student Thinking (Consolidate)* Invite students to make a sound mathematical decision about the meaning and usefulness of new or deepened learning by connecting it back to their thinking at the start of the lesson and making appropriate extensions and revisions to improve their initial response (reflective reasoning).

Empowering Students with Reflective Reasoning

Central to student success in math is the provision of rich learning opportunities supported by the capacity to reason reflectively. Students' reasoning processes must be iterative so that they can simultaneously develop, deepen, and automatize an understanding of foundational math concepts and content as well as their reasoning competencies. This can be achieved at a lesson level using a *reasoning-focused* three-part lesson structure, as outlined in Table 3.

GUIDING PRINCIPLES FOR POWERFUL LEARNING IN MATH

A set of six fundamental learning principles underpin this reasoning approach to math learning. They describe the nature of the power of this pedagogy to empower all learners.

1. **Balanced:** Students learn math content while simultaneously learning *how* to reason proficiently. There is a natural interplay between the learning of math computational, procedural, and algorithmic foundations and the development of mathematical reasoning competencies.
2. **Comprehensive:** Students explicitly and intentionally learn to develop proficiency across a full range of mathematical reasoning competencies, with a focus on sound reasoning to understand, act, and communicate mathematically.
3. **Sustained:** Student learning is iterative,

cultivated, and automatized by focusing on the proficient development of two core reasoning competencies: sound reasoning and reflective reasoning.

4. **Authentic:** Student learning is framed within mathematically meaningful problematic contexts with an authentic purpose and/or audience to support the development of practical mathematical intelligence for everyday, everywhere use.
5. **Self-Regulated:** Student learning is designed to help students become strong, independent, self-regulating, self-correcting, reflective learners who appreciate the productive power of not getting everything perfect right away and the potential value of collaboration.
6. **Naturally Self-Motivating:** Student learning is designed to allow students to begin by anticipating what and how they need to learn and by appreciating and trusting that multiple learning opportunities focused on clear learning goals will support this learning. This ensures that *every* lesson has built-in relevance for students as it clearly and continuously connects back to meaningful learning goals.

At the core of all of these principles is *student learning empowered by student reasoning*. Students learn how to understand and select knowledge and how to act and communicate mathematically using reasoning competencies. These guiding principles help educators make instructional and assessment decisions about when and how to support student learning and thinking in math. In addition, they consolidate and extend the benefits of current diverse approaches to the teaching and learning of mathematics.

The Benefits of This Approach

When time is invested in helping students learn how to reason mathematically through critical thinking, they become increasingly confident, self-aware, flexible, and fluid problem managers who approach math with persistence, resilience, and enthusiasm. They come to understand the value of math in their lives and their world while learning to think with clarity, precision, and depth of understanding.

These benefits accrue as a result of a number of advantages inherent to an approach to learning based on reasoning competency and critical thinking. These advantages include

1. a reconciliation of deep conceptual understanding with automaticity of knowledge,
2. a sustained focus on long-term enduring learning success rather than on short-term temporary gains,
3. the fostering of creative and innovative thinking that results from rich iterative learning experiences,
4. the use of collaboration to grow individual mathematical understanding and reasoning competencies beyond what was originally possible, and
5. the cultivation of other thinking literacies.

No matter the purpose, context, or nature of the mathematical study we wish students to pursue, the proficient development of mathematical reasoning competencies is at the core of sustained, successful student learning in mathematics and other related fields. In the absence of sound reasoning through critical thinking in math, student success will be limited to recall, reproduction, or creation of solutions that fit their preferences. The time has come to focus math education on reasoning competencies that will permit students to construct deep and meaningful understandings about mathematical truths, help them discover new truths, and lead them to make new and insightful creative connections.

APPENDIX

Sources Used to Develop a Synthesized Set of Mathematical Reasoning Competencies

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